

# Digital Communication Systems

## EES 452

Asst. Prof. Dr. Prapun Suksompong

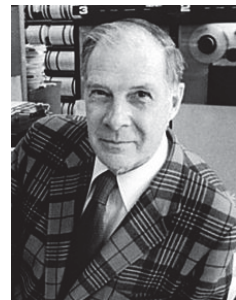
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### 5.1 Binary Linear Block Codes

## Introduction to Hamming Code

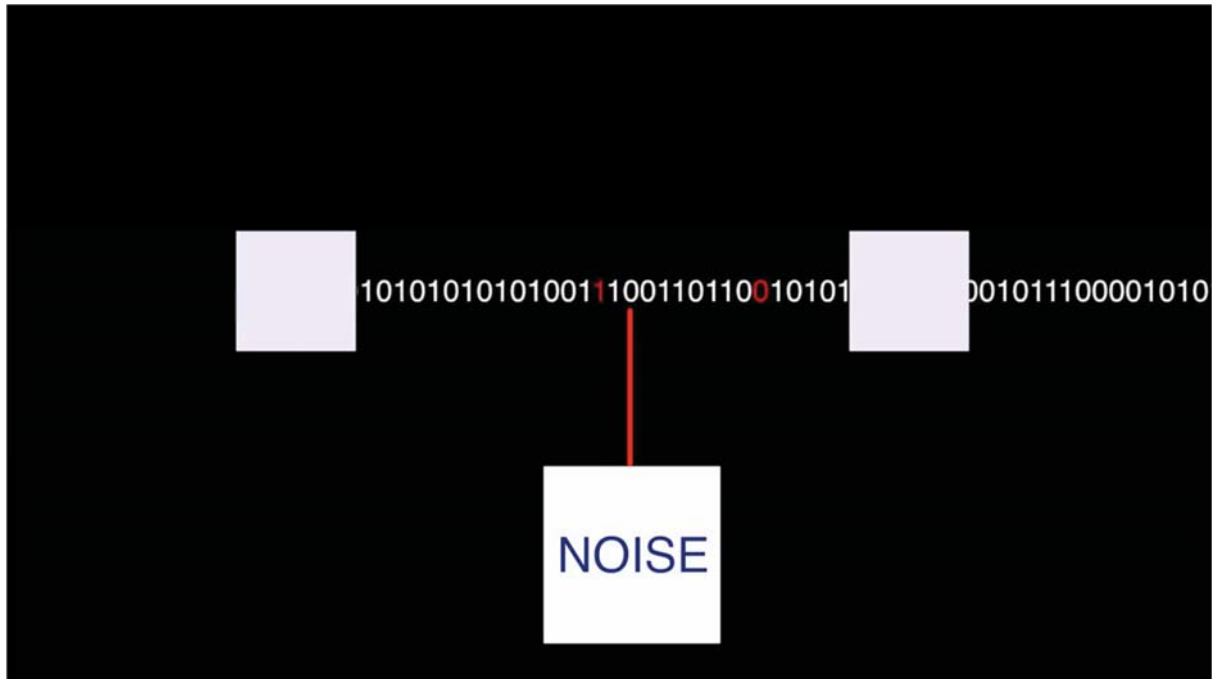
### Hamming codes

- One of the earliest codes studied in coding theory.
- Named after Richard W. Hamming
  - The IEEE Richard W. **Hamming Medal**, named after him, is an award given annually by Institute of Electrical and Electronics Engineers (IEEE), for "exceptional contributions to information sciences, systems and technology".
    - Sponsored by Qualcomm, Inc
    - Some Recipients:
      - 1988 - Richard W. Hamming
      - 1997 - Thomas M. Cover
      - 1999 - David A. Huffman
      - 2011 - Toby Berger
- The simplest of a class of (algebraic) error correcting codes that **can correct one error in a block of bits**





# Hamming codes

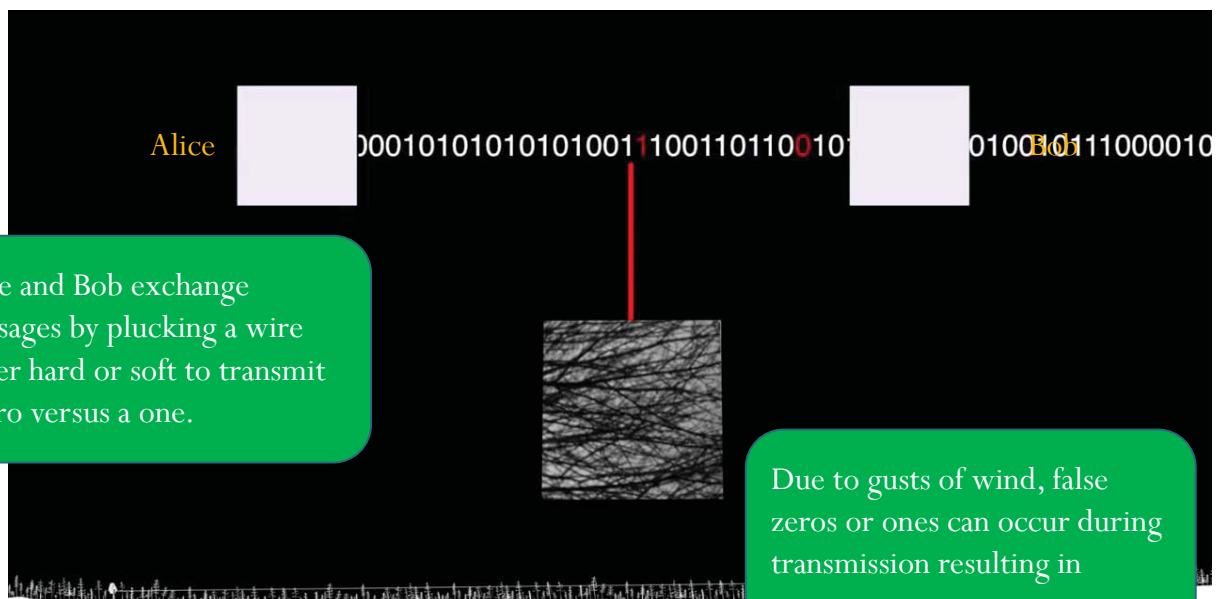


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[<https://www.youtube.com/watch?v=cBBTWcHkVVY>]



# Hamming codes



Alice and Bob exchange messages by plucking a wire either hard or soft to transmit a zero versus a one.

Due to gusts of wind, false zeros or ones can occur during transmission resulting in errors.

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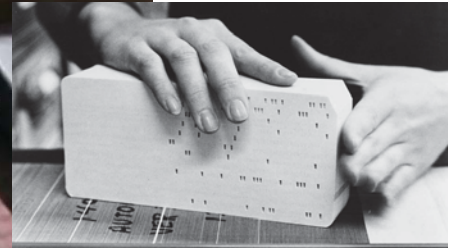
In the 1940s, Richard Hamming faced a similar problem while working at Bell Laboratories.



# Hamming codes



At the time the computers used stored information on punch cards representing one versus zero with hole versus no hole.



This system was error-prone because it was common for cards to get bent or miss punched in the first place so holes could be missed or no holes could be accidentally punctured causing flipped bits.

Hamming took it upon himself to devise a method which could automatically detect and correct single bit errors without interrupting calculations.

# Hamming codes

## The Bell System Technical Journal

Vol. XX<sup>IX</sup>

April, 1950

No. 2

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### Error Detecting and Error Correcting Codes

By R. W. HAMMING

#### 1. INTRODUCTION

**T**HE author was led to the study given in this paper from a consideration of large scale computing machines in which a large number of operations must be performed without a single error in the end result. This problem of "doing things right" on a large scale is not essentially new; in a



# Hamming codes

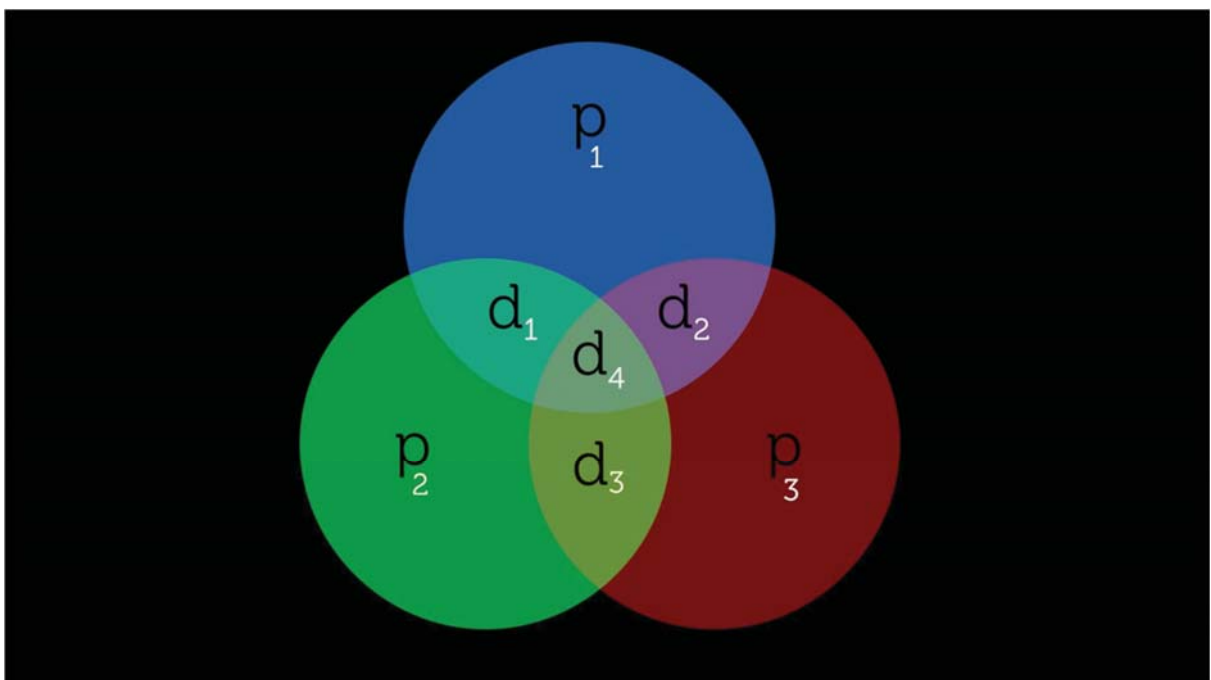


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[<https://www.youtube.com/watch?v=cBBTWcHkVVY>]



# Hamming codes: Ex. 1



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[<https://www.youtube.com/watch?v=cBBTWcHkVVY>]

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### 5.1 Binary Linear Block Codes

## Code Structure and the Generator Matrix



### Review: Even Parity

- A binary vector (or a collection of 1s and 0s) has **even parity** if and only if the number of 1s in there is even.
  - Suppose we are given the values of all the bits except one bit.
    - We can force the vector to have even parity by setting the value of the remaining bit to be the sum of the other bits.

Single-parity-check code

[1 0 1 1 0 \_]

Square array

1	0	1	_
0	1	1	_
0	0	1	_
-	-	-	-

# Review: Even Parity

- A binary vector (or a collection of 1s and 0s) has **even parity** if and only if the number of 1s in there is even.
  - Suppose we are given the values of all the bits except one bit.
    - We can force the vector to have even parity by setting the value of the remaining bit to be the sum of the other bits.

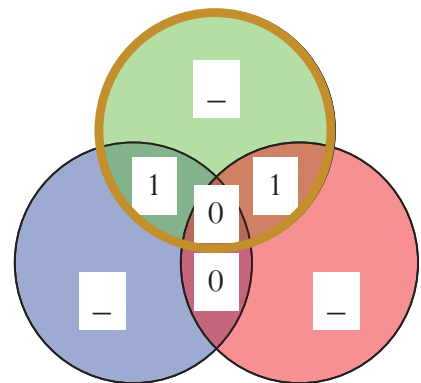
Single-parity-check code

[1 0 1 1 0 \_]

Square array

1	0	1	_
0	1	1	_
0	0	1	_

Hamming code



# Hamming codes: Ex. 1

This is an example of Hamming (7,4) code

In the video, the codeword is constructed from the data by

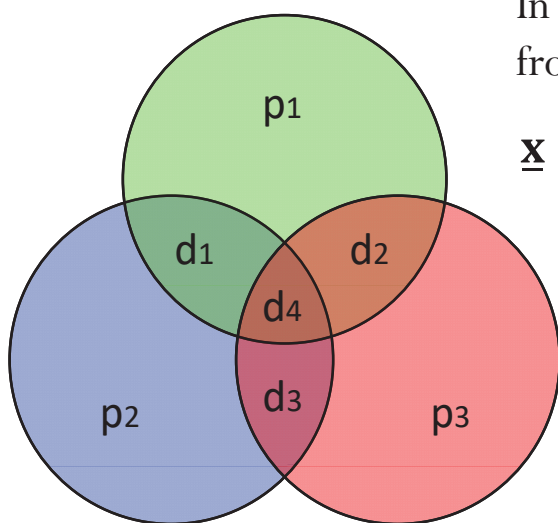
$$\underline{x} = [p_1 \ d_1 \ p_2 \ d_2 \ p_3 \ d_3 \ d_4]$$

where

$$p_1 = d_1 \oplus d_2 \oplus d_4$$

$$p_2 = d_1 \oplus d_3 \oplus d_4$$

$$p_3 = d_2 \oplus d_3 \oplus d_4$$



- The **message bits** are also referred to as the **data bits** or **information bits**.
- The **non-message bits** are also referred to as **parity check bits**, **checksum bits**, **parity bits**, or **check bits**.

Writing the generator matrix from the code “structure”

## Generator matrix: a revisit

- Fact: The 1s and 0s in the  $j^{\text{th}}$  column of  $\mathbf{G}$  tells which positions of the data bits are combined ( $\oplus$ ) to produce the  $j^{\text{th}}$  bit in the codeword.
- For the Hamming code in the previous slide,

$$\underline{\mathbf{x}} = [p_1 \quad d_1 \quad p_2 \quad d_2 \quad p_3 \quad d_3 \quad d_4]$$

$$\begin{aligned} p_1 &= d_1 \oplus d_2 \oplus d_4 \\ p_2 &= d_1 \oplus d_3 \oplus d_4 \\ p_3 &= d_2 \oplus d_3 \oplus d_4 \end{aligned}$$

$$= [d_1 \quad d_2 \quad d_3 \quad d_4] \underbrace{\begin{bmatrix} 1 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ 0 & 0 & & & & & & \\ 1 & 0 & & & & & & \end{bmatrix}}_{\mathbf{G}}$$

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“Reading” the generator matrix from the codebook.

	$\underline{\mathbf{d}}$	$\underline{\mathbf{x}}$
	0 0 0 0	0 0 0 0 0 0 0
$d_4$	0 0 0 1	1 0 1 0 1 0 1
$d_3$	0 0 1 0	0 0 1 0 1 1 0
	0 0 1 1	1 0 0 0 0 1 1
$d_2$	0 1 0 0	1 0 0 1 1 0 0
	0 1 0 1	0 0 1 1 0 0 1
	0 1 1 0	1 0 1 1 0 1 0
	0 1 1 1	0 0 0 1 1 1 1
$d_1$	1 0 0 0	1 1 1 0 0 0 0
	1 0 0 1	0 1 0 0 1 0 1
	1 0 1 0	1 1 0 0 1 1 0
	1 0 1 1	0 1 1 0 0 1 1
	1 1 0 0	0 1 1 1 1 0 0
	1 1 0 1	1 1 0 1 0 0 1
	1 1 1 0	0 1 0 1 0 1 0
	1 1 1 1	1 1 1 1 1 1 1

- One can also “read”  $\mathbf{G}$  from the codebook.

- From  $\underline{\mathbf{x}} = \underline{\mathbf{d}}\mathbf{G} =$

$$\sum_{j=1}^k d_j \underline{\mathbf{g}}^{(j)},$$

when we look at the message block with a single 1 at position  $i$ , then the corresponding codeword is the same as  $\underline{\mathbf{g}}^{(i)}$ .

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} = \mathbf{G}$$

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# Checking whether a code is generated by some generator matrix $G$

$\underline{d}$	$\underline{x}$
0 0 0 0	0 0 0 0 0 0 0
0 0 0 1	1 0 1 0 1 0 1
0 0 1 0	0 0 1 0 1 1 0
0 0 1 1	1 0 0 0 0 1 1
0 1 0 0	1 0 0 1 1 0 0
0 1 0 1	0 0 1 1 0 0 1
0 1 1 0	1 0 1 1 0 1 0
0 1 1 1	0 0 0 1 1 1 1
1 0 0 0	1 1 1 0 0 0 0
1 0 0 1	0 1 0 0 1 0 1
1 0 1 0	1 1 0 0 1 1 0
1 0 1 1	0 1 1 0 0 1 1
1 1 0 0	0 1 1 1 1 0 0
1 1 0 1	1 1 0 1 0 0 1
1 1 1 0	0 1 0 1 0 1 0
1 1 1 1	1 1 1 1 1 1 1

- If a code is generated by a generator matrix, it is automatically a linear code.
- When the codebook is provided, look at each column of the codeword part.
- Write down the equation by reading the structure from appropriate row discussed earlier.
  - For example, here, we read  $x_1 = d_1 \oplus d_2 \oplus d_4$ .
- Then, we add the corresponding columns of the message part and check whether the sum is the same as the corresponding codeword column.
- So, we need to check  $n$  summations.
  - Direct checking that we discussed earlier consider  $\binom{M-1}{2}$  summations.

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## Example:

$\underline{d}$	$\underline{x}$
0 0 0 0	0 0 0 0 0 0 0
0 0 0 1	1 0 1 0 1 0 1
0 0 1 0	0 0 1 0 1 1 0
0 0 1 1	1 0 0 0 0 1 1
0 1 0 0	1 0 0 1 1 0 0
0 1 0 1	0 0 1 1 0 0 1
0 1 1 0	1 0 1 1 0 1 0
0 1 1 1	0 0 0 1 1 1 1
1 0 0 0	1 1 1 0 0 0 0
1 0 0 1	0 1 0 0 1 0 1
1 0 1 0	1 1 0 0 1 1 0
1 0 1 1	0 1 1 0 0 1 1
1 1 0 0	1 1 1 1 1 0 0
1 1 0 1	1 1 0 1 0 0 1
1 1 1 0	0 1 0 1 0 1 0
1 1 1 1	1 1 1 1 1 1 1

- We read  $x_1 = d_1 \oplus d_2 \oplus d_4$ .
- We add the message columns corresponding to  $d_1, d_2, d_4$ ,
  - We see that the first bit of the 13<sup>th</sup> codeword does not conform with the structure above.
- Conclusion: This code is not generated by a generator matrix.

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## Example:

<u>d</u>	<u>x</u>
0 0 0 0	0 0 0 0 0 0 0 0
0 0 0 1	1 0 1 0 1 0 1
0 0 1 0	0 0 1 0 1 1 0
0 0 1 1	1 0 0 0 0 1 1
0 1 0 0	1 0 0 1 1 0 0
0 1 0 1	0 0 1 1 0 0 1
0 1 1 0	1 0 1 1 0 1 0
0 1 1 1	0 0 0 1 1 1 1
1 0 0 0	1 1 1 0 0 0 0
1 0 0 1	0 1 0 0 1 0 1
1 0 1 0	1 1 0 0 1 1 0
1 0 1 1	0 1 1 0 0 1 1
1 1 0 0	1 1 1 1 1 0 0
1 1 0 1	1 1 0 1 0 0 1
1 1 1 0	0 1 0 1 0 1 0
1 1 1 1	1 1 1 1 1 1 1

- The case found in the previous slide may help with the search to show that the code is not linear.

- The corresponding message is 1100.

- The codeword corresponding to this message should be  $\underline{\mathbf{g}}^{(1)} \oplus \underline{\mathbf{g}}^{(2)}$ .

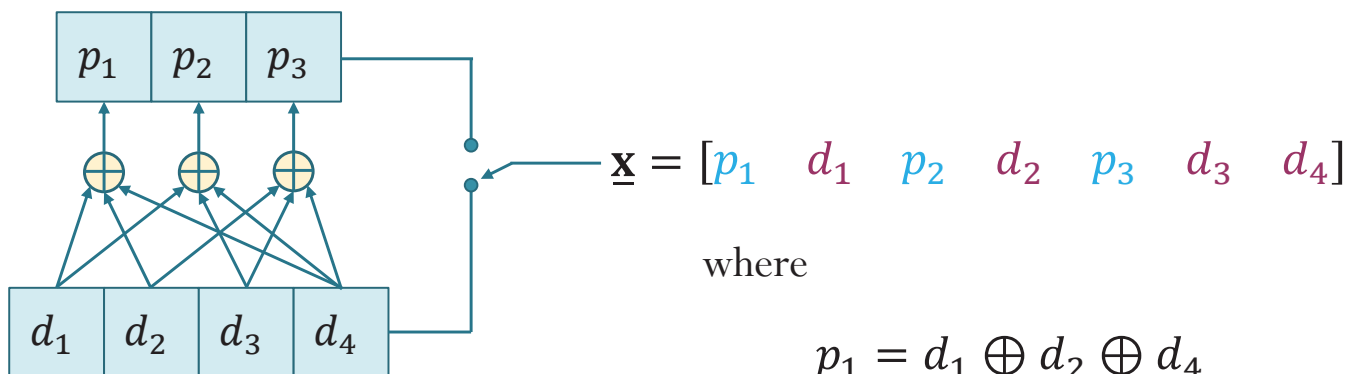
- If  $\underline{\mathbf{g}}^{(1)} \oplus \underline{\mathbf{g}}^{(2)}$ , is not a codeword, then we can quickly conclude that the code is not linear:

$\underline{\mathbf{g}}^{(1)}$  and  $\underline{\mathbf{g}}^{(2)}$  are codewords but  $\underline{\mathbf{g}}^{(1)} \oplus \underline{\mathbf{g}}^{(2)} = 0111100$  is not one of the codewords.

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## Implementation

- Linear block codes are typically implemented with modulo-2 adders tied to the appropriate stages of a shift register.



where

$$p_1 = d_1 \oplus d_2 \oplus d_4$$

$$p_2 = d_1 \oplus d_3 \oplus d_4$$

$$p_3 = d_2 \oplus d_3 \oplus d_4$$

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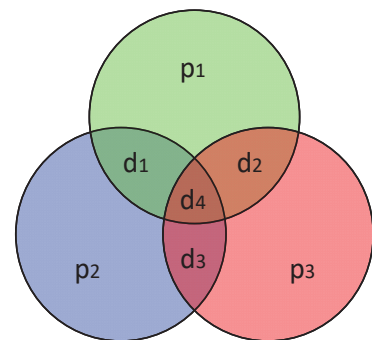
### 5.1 Binary Linear Block Codes

## Parity-Check Matrix H

Back to

### Hamming codes: Ex. 1

$$\underline{\mathbf{x}} = [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7] = [p_1 \ d_1 \ p_2 \ d_2 \ p_3 \ d_3 \ d_4]$$



Structure in the codewords:

$$\begin{aligned} p_1 &= d_1 \oplus d_2 \oplus d_4 & \iff & \quad p_1 \oplus d_1 \oplus d_2 \oplus d_4 = 0 \\ p_2 &= d_1 \oplus d_3 \oplus d_4 & & \quad p_2 \oplus d_1 \oplus d_3 \oplus d_4 = 0 \\ p_3 &= d_2 \oplus d_3 \oplus d_4 & & \quad p_3 \oplus d_2 \oplus d_3 \oplus d_4 = 0 \end{aligned}$$

At the receiver, we check whether the received vector  $\underline{\mathbf{y}}$  still satisfies these conditions via computing the **syndrome vector**:

$$\underline{\mathbf{s}} = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \underline{\mathbf{0}}?$$

# Parity Check Matrix: Ex 1

- Intuitively, the **parity check matrix  $\mathbf{H}$** , as the name suggests, tells which bits in the observed vector  $\underline{y}$  are used to “check” for validity of  $\underline{y}$ .
- The number of rows is the same as the number of conditions to check (which is the same as the number of parity check bits).
- For each row, a one indicates that the bits (including the bits in the parity positions) are used in the validity check calculation.

Structure in the codeword:

$$p_1 \oplus d_1 \oplus d_2 \oplus d_4 = 0$$

$$p_2 \oplus d_1 \oplus d_3 \oplus d_4 = 0$$

$$p_3 \oplus d_2 \oplus d_3 \oplus d_4 = 0$$



$$\mathbf{H} = \begin{matrix} & \begin{matrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

# Parity Check Matrix: Ex 1

Relationship between  $\mathbf{G}$  and  $\mathbf{H}$ .

$$\mathbf{G} = \begin{matrix} & \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{matrix} \iff \mathbf{H} = \begin{matrix} & \begin{matrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$



# Parity Check Matrix: Ex 1

Relationship between **G** and **H**.

$$\begin{array}{cccccc}
 x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
 p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4
 \end{array}$$

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\longleftrightarrow$$

$$\begin{array}{cccccc}
 y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\
 x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
 p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4
 \end{array}$$

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$



# Parity Check Matrix: Ex 1

Relationship between **G** and **H**.

$$\begin{array}{cccccc}
 x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
 p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4
 \end{array}$$

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\longleftrightarrow$$

$$\begin{array}{cccccc}
 y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\
 x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
 p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4
 \end{array}$$

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(columns of) identity matrix  
in the data positions

(columns of) identity matrix  
in the parity check positions



# Parity Check Matrix: Ex 1

Relationship between **G** and **H**.

$$\mathbf{G} = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \end{array} \\ \left[ \begin{array}{ccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] \end{array} \longleftrightarrow \mathbf{H} = \begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \end{array} \\ \left[ \begin{array}{ccccccc} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right] \end{array}$$

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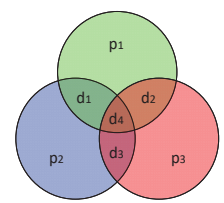
[prapun@siit.tu.ac.th](mailto:prapun@siit.tu.ac.th)

### 5.1 Binary Linear Block Codes

More on H and G,  
and Systematic Encoding

# Review: Linear Block Codes

Code structure  
 $\underline{x} = [p_1 \ d_1 \ p_2 \ d_2 \ p_3 \ d_3 \ d_4]$   
 where  $p_1 = d_1 \oplus d_2 \oplus d_4$   
 $p_2 = d_1 \oplus d_3 \oplus d_4$   
 $p_3 = d_2 \oplus d_3 \oplus d_4$



Codebook

$\underline{d}$	$\underline{x}$
0 0 0 0	0 0 0 0 0 0 0
0 0 0 1	1 0 1 0 1 0 1
0 0 1 0	0 0 1 0 1 1 0
0 0 1 1	1 0 0 0 0 1 1
0 1 0 0	1 0 0 1 1 0 0
0 1 0 1	0 0 1 1 0 0 1
0 1 1 0	1 0 1 1 0 1 0
0 1 1 1	0 0 0 1 1 1 1
1 0 0 0	1 1 1 0 0 0 0
1 0 0 1	0 1 0 0 1 0 1
1 0 1 0	1 1 0 0 1 1 0
1 0 1 1	0 1 1 0 0 1 1
1 1 0 0	0 1 1 1 1 0 0
1 1 0 1	1 1 0 1 0 0 1
1 1 1 0	0 1 0 1 0 1 0
1 1 1 1	1 1 1 1 1 1 1

$$G = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Generator Matrix

$$H = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Parity Check Matrix



# Review: Linear Block Codes

- The code structure is **built** into each codeword at the encoder (transmitter) via the generator matrix
  - Each codeword is created by  $\underline{x} = \underline{d}G$ .
- The code structure is **checked** at the decoder (receiver) via the parity check matrix.
  - A valid codeword must satisfy  $\underline{x}H^T = \underline{0}$ .

$$G = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \longleftrightarrow H = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

## Remark: Location of the Message Bits

$$\mathbf{G} = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \longleftrightarrow \mathbf{H} = \begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

- The “identity-matrix” columns in  $\mathbf{G}$  corresponds to positions of the **message (data) bits** in each codeword.
  - Ex. For this code, codeword  $\underline{\mathbf{x}} = [1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0]$  corresponds to message  $\underline{\mathbf{b}} = [1 \ 0 \ 1 \ 0]$ .
- The “identity-matrix” columns in  $\mathbf{H}$  corresponds to positions of the parity (check) bits in each codeword.

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## Parity Check Matrix

Key property:

$$\mathbf{GH}^T = \mathbf{0}_{k \times (n-k)}$$

Proof:

- When there is no error ( $\underline{\mathbf{e}} = \underline{\mathbf{0}}$ ), the syndrome vector calculation should give  $\underline{\mathbf{s}} = \underline{\mathbf{0}}$ .

- By definition,

$$\underline{\mathbf{s}} = \underline{\mathbf{y}}\mathbf{H}^T = (\underline{\mathbf{x}} \oplus \underline{\mathbf{e}})\mathbf{H}^T = \underline{\mathbf{x}}\mathbf{H}^T \oplus \underline{\mathbf{e}}\mathbf{H}^T = \underline{\mathbf{b}}\mathbf{GH}^T \oplus \underline{\mathbf{e}}\mathbf{H}^T.$$

- Therefore, when  $\underline{\mathbf{e}} = \underline{\mathbf{0}}$ , we have  $\underline{\mathbf{s}} = \underline{\mathbf{b}}\mathbf{GH}^T$ .

- To have  $\underline{\mathbf{s}} = \underline{\mathbf{0}}$  for any  $\underline{\mathbf{b}}$ , we must have  $\mathbf{GH}^T = \mathbf{0}$ .

A matrix of zeroes

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# Systematic Encoding

- Code constructed with distinct information bits and check bits in each codeword are called **systematic codes**.
  - **Message bits** are “visible” in the codeword.
- Popular forms of  $\mathbf{G}$ :

$$\mathbf{G} = \left[ \mathbf{P}_{k \times (n-k)} \mid \mathbf{I}_k \right] \quad \underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = \begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} \left[ \mathbf{P}_{k \times (n-k)} \mid \mathbf{I}_k \right]$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_{n-k} & \mid & b_1 & b_2 & \cdots & b_k \end{bmatrix}$$

$x_{n-k+1}$     $x_{n-k+2}$     $x_n$

$$\mathbf{G} = \left[ \mathbf{I}_k \mid \mathbf{P}_{k \times (n-k)} \right] \quad \underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = \begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} \left[ \mathbf{I}_k \mid \mathbf{P}_{k \times (n-k)} \right]$$

$$= \begin{bmatrix} b_1 & b_2 & \cdots & b_k & \mid & x_{k+1} & x_{k+2} & \cdots & x_n \end{bmatrix}$$

$x_1$     $x_2$     $x_k$

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# Parity check matrix

- For the generators matrices we discussed in the previous slide, the corresponding **parity check matrix** can be found easily:

$$\mathbf{G} = \left[ \mathbf{P}_{k \times (n-k)} \mid \mathbf{I}_k \right] \quad \longrightarrow \quad \mathbf{H} = \left[ \mathbf{I}_{n-k} \mid -\mathbf{P}^T \right]$$

$$\text{Check: } \mathbf{GH}^T = \left[ \mathbf{P} \mid \mathbf{I} \right] \begin{bmatrix} \mathbf{I} \\ -\mathbf{P} \end{bmatrix} = \mathbf{P} \oplus (-\mathbf{P}) = \mathbf{0}_{k \times (n-k)}$$

$$\mathbf{G} = \left[ \mathbf{I}_k \mid \mathbf{P}_{k \times (n-k)} \right] \quad \longrightarrow \quad \mathbf{H} = \left[ -\mathbf{P}^T \mid \mathbf{I}_{n-k} \right]$$

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## Hamming codes: Ex. 2

- Systematic (7,4) Hamming Codes

$$\mathbf{G} = \left[ \begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\mathbf{H} = \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array} \right]$$

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# Digital Communication Systems EES 452

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5.1 Binary Linear Block Codes

Formal Construction of  
Hamming Code

# Hamming codes

Now, we will give a general recipe for constructing Hamming codes.

Parameters:

- $m = n - k =$  number of parity bits
- $n = 2^m - 1 \in \{3, 7, 15, 31, 63, 127, \dots\}$
- $k = n - m = 2^m - m - 1$

It can be shown that, for Hamming codes,

- $d_{\min} = 3$ .
- Error correcting capability:  $t = 1$

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# Construction of Hamming Codes

- Start with  $m$ .
1. Parity check matrix  $\mathbf{H}$ :
    - Construct a matrix whose columns consist of *all* nonzero binary  $m$ -tuples.
    - The ordering of the columns is arbitrary.  
However, next step is easy when the columns are arranged so that  $\mathbf{H} = [\mathbf{I}_m \mid \mathbf{P}]$ .
  2. Generator matrix  $\mathbf{G}$ :
    - When  $\mathbf{H} = [\mathbf{I}_m \mid \mathbf{P}]$ , we have  $\mathbf{G} = [-\mathbf{P}^T \mid \mathbf{I}_k] = [\mathbf{P}^T \mid \mathbf{I}_k]$ .

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# Hamming codes: Ex. 2

- Systematic (7,4) Hamming Codes

$$\mathbf{H} = \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array} \right] \begin{array}{l} \\ \\ -\mathbf{P}^T \end{array}$$

↓

$$\mathbf{G} = \left[ \begin{array}{ccc|cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ \\ \mathbf{P} \\ \end{array}$$

- Columns are all possible nonzero 3-bit vectors
- We arrange the columns so that  $\mathbf{I}_3$  is on the left to make the code systematic. (One can also put  $\mathbf{I}_3$  on the right.)
- Note that the size of the identity matrices in  $\mathbf{G}$  and  $\mathbf{H}$  are not the same.

# Review: Hamming Code Recipe

Here,  
 $m = 3$   
 $n = 2^3 - 1 = 7$   
 $k = 4$

$$\mathbf{H} = \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array} \right] \begin{array}{l} \mathbf{I}_m \\ \mathbf{P}^T \end{array}$$

↙ ↘

$$\mathbf{G} = \left[ \begin{array}{ccc|cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ \mathbf{P} \\ \mathbf{I}_k \end{array}$$

- Start with the parity-check matrix
- $m$  rows
  - $m = n - k$
- Columns are all possible nonzero  $m$ -bit vectors
  - $n = 2^m = 1$  columns
  - Arranged to have  $\mathbf{I}_m$  on the left (or on the right).
    - This simplifies conversion to  $\mathbf{G}$ .
- Get  $\mathbf{G}$  from  $\mathbf{H}$ .

$$\mathbf{G} = \left[ \mathbf{P}_{k \times (n-k)} \mid \mathbf{I}_k \right] \longleftrightarrow \mathbf{H} = \left[ \mathbf{I}_{n-k} \mid -\mathbf{P}^T \right]$$

- Note that the size of the identity matrices in  $\mathbf{G}$  and  $\mathbf{H}$  can be different.

# Digital Communication Systems

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### 5.1 Binary Linear Block Codes

## Properties of Syndrome Vector and Decoding of Linear Codes

### Minimum Distance Decoding

- At the decoder, suppose we want to use minimum distance decoding, then
  - The decoder needs to have the list of all the possible codewords so that it can compare their distances to the received vector  $\underline{\mathbf{y}}$ .
  - There are  $2^k$  codewords each having  $n$  bits. Therefore, saving these takes  $2^k \times n$  bits.
  - Also, we will need to perform the comparison  $2^k$  times.
- Alternatively, we can utilize the syndrome vector.
  - The syndrome vector is computed from the parity-check matrix  $\mathbf{H}$ .
  - Therefore, saving  $\mathbf{H}$  takes  $(n - k) \times n$  bits.

# Minimum Distance Decoding

- Recall that

$$d(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = w(\underline{\mathbf{x}} \oplus \underline{\mathbf{y}}) = w(\underline{\mathbf{e}})$$

- Therefore, minimizing the distance is the same as minimizing the weight of the error pattern.
- New goal:
  - find the decoded error pattern  $\hat{\underline{\mathbf{e}}}$  with the minimum weight
  - then, the decoded codeword is  $\hat{\underline{\mathbf{x}}} = \underline{\mathbf{y}} \oplus \hat{\underline{\mathbf{e}}}$
- Once we know  $\hat{\underline{\mathbf{x}}}$ , we can directly extract the message part from the decoded codeword if we are using systematic code.
- For example, consider

$$\mathbf{G} = \left[ \begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

Suppose  $\hat{\underline{\mathbf{x}}} = 1011010$ , then we know that the decoded message is  $\hat{\underline{\mathbf{b}}} = 1010$ .

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# Properties of Syndrome Vector

- Recall that, from  $\mathbf{GH}^T = \mathbf{0}$ , we have

$$\underline{\mathbf{s}} = \underline{\mathbf{y}}\mathbf{H}^T = (\underline{\mathbf{x}} \oplus \underline{\mathbf{e}})\mathbf{H}^T = (\underline{\mathbf{b}}\mathbf{G} \oplus \underline{\mathbf{e}})\mathbf{H}^T = \underline{\mathbf{e}}\mathbf{H}^T$$

- Thinking of  $\mathbf{H}$  as a matrix with many columns inside,

$$\mathbf{H} = \left[ \begin{array}{c} \underline{\mathbf{h}}^{(1)} \\ \underline{\mathbf{h}}^{(2)} \\ \vdots \\ \underline{\mathbf{h}}^{(n-k)} \end{array} \right]_{(n-k) \times n} = \left[ \begin{array}{c} \left( \underline{\mathbf{v}}^{(1)} \right)^T \\ \left( \underline{\mathbf{v}}^{(2)} \right)^T \\ \dots \\ \left( \underline{\mathbf{v}}^{(n)} \right)^T \end{array} \right]$$

$$\underline{\mathbf{s}} = \underline{\mathbf{e}}\mathbf{H}^T = \sum_{j=1}^n e_j \underline{\mathbf{v}}^{(j)}$$

- Therefore,  $\underline{\mathbf{s}}$  is a (linear combination of the columns of  $\mathbf{H}$ )<sup>T</sup>.

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## Hamming Codes: Ex. 2

$$\mathbf{H} = \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array} \right]$$

$$\underline{\mathbf{s}} = \underline{\mathbf{e}}\mathbf{H}^T = \sum_{j=1}^n e_j \underline{\mathbf{v}}^{(j)}$$

Linear  
 combination of  
 the columns of  $\mathbf{H}$

Note that for an error pattern with a single one in the  $j^{\text{th}}$  coordinate position, the syndrome  $\underline{\mathbf{s}} = \underline{\mathbf{y}}\mathbf{H}^T$  is the same as the  $j^{\text{th}}$  column of  $\mathbf{H}$ .

Error pattern $\underline{\mathbf{e}}$	Syndrome = $\underline{\mathbf{e}}\mathbf{H}^T$
(0,0,0,0,0,0,0)	(0,0,0)
(0,0,0,0,0,0,1)	(1,1,1)
(0,0,0,0,0,1,0)	(1,1,0)
(0,0,0,0,1,0,0)	(1,0,1)
(0,0,0,1,0,0,0)	(0,1,1)
(0,0,1,0,0,0,0)	(0,0,1)
(0,1,0,0,0,0,0)	(0,1,0)
(1,0,0,0,0,0,0)	(1,0,0)

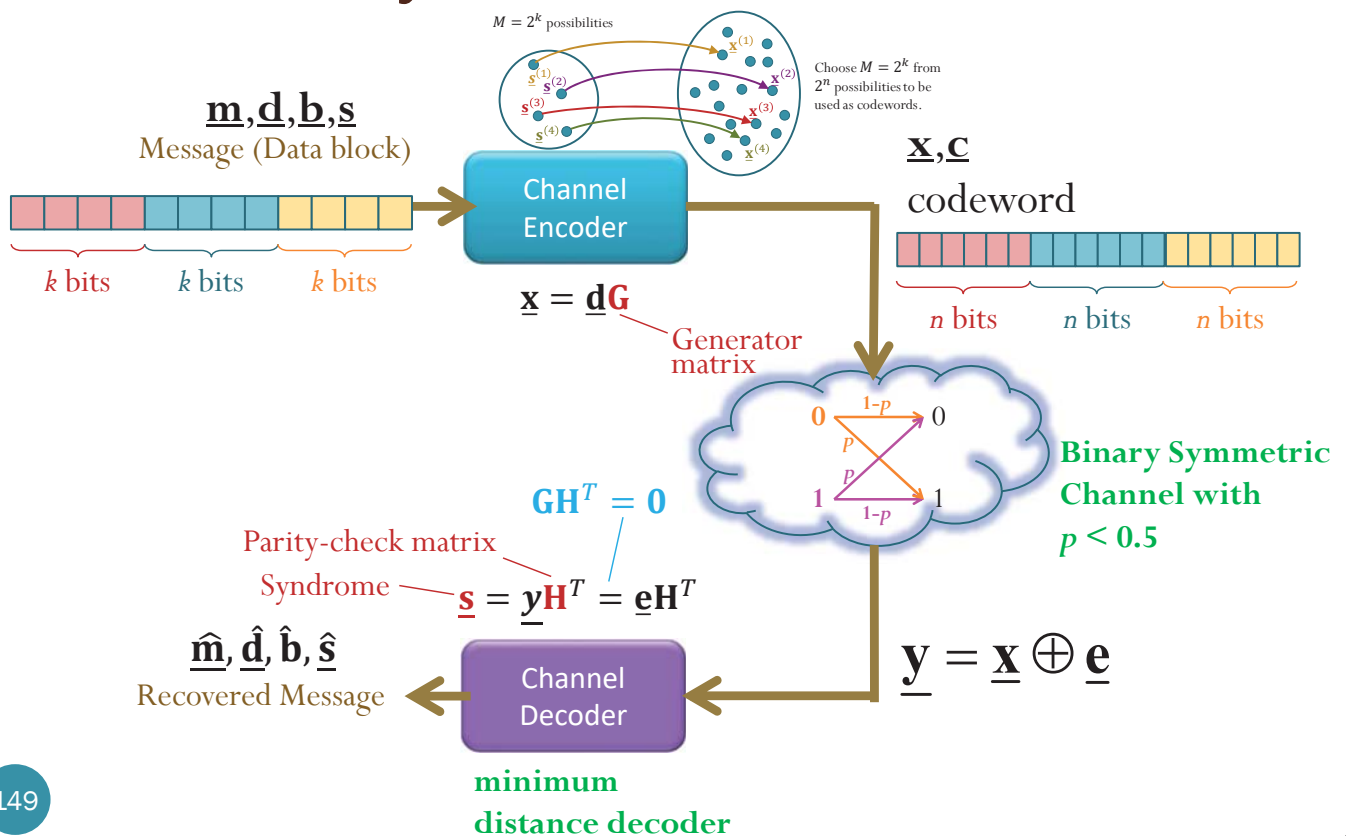
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## Decoding Algorithm

- Assumption: the columns of  $\mathbf{H}$  are nonzero and distinct.
- Compute the **syndrome**  $\underline{\mathbf{s}} = \underline{\mathbf{y}}\mathbf{H}^T$  for the received vector.
- Case 1: If  $\underline{\mathbf{s}} = \underline{\mathbf{0}}$ , set  $\underline{\hat{\mathbf{x}}} = \underline{\mathbf{y}}$ .
- Case 2: If  $\underline{\mathbf{s}} \neq \underline{\mathbf{0}}$ ,
  - determine the position  $j$  of the column of  $\mathbf{H}$  that is the same as (the transposition) of the syndrome,
  - set  $\underline{\hat{\mathbf{x}}} = \underline{\mathbf{y}}$  but with the  $j^{\text{th}}$  bit complemented.
- For Hamming codes, because the columns are constructed from all possible non-zero  $m$ -tuples, the syndrome vectors must fall into one of the two cases considered above.
- For general linear block codes, the two cases above may not cover every cases.

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# Summary: Linear Block Code



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## Hamming Codes: Ex. 1

- Consider the Hamming code with

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \iff \mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

- Suppose we observe  $\underline{y} = [0 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1]$  at the receiver. Find the decoded codeword and the decoded message.

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[To be explored in the HW]

## Hamming Codes: The original method

- Encoding
  - The bit positions that are powers of 2 (1, 2, 4, 8, 16, etc.) are check bits.
  - The rest (3, 5, 6, 7, 9, etc.) are filled up with the  $k$  data bits.
  - Each check bit forces the parity of some collection of bits, including itself, to be even (or odd).
    - To see which check bits the data bit in position  $i$  contributes to, rewrite  $i$  as a sum of powers of 2. A bit is checked by just those check bits occurring in its expansion
- Decoding
  - When a codeword arrives, the receiver initializes a counter to zero. It then examines each check bit at position  $i$  ( $i = 1, 2, 4, 8, \dots$ ) to see if it has the correct parity.
  - If not, the receiver adds  $i$  to the counter. If the counter is zero after all the check bits have been examined (i.e., if they were all correct), the codeword is accepted as valid. If the counter is nonzero, it contains the position of the incorrect bit.

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# Digital Communication Systems

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### 5.1 Binary Linear Block Codes

## Proof of the Decoding Algorithm



# Proof of the Decoding Algorithm

- We will assume that the columns of  $\mathbf{H}$  are nonzero and distinct.
  - This is automatically satisfied for Hamming codes constructed from our recipe.
- Case 1: When  $\underline{\mathbf{e}} = \underline{\mathbf{0}}$ , we have  $\underline{\mathbf{s}} = \underline{\mathbf{0}}$ .
  - When  $\underline{\mathbf{s}} = \underline{\mathbf{0}}$ , we can conclude that  $\hat{\underline{\mathbf{e}}} = \underline{\mathbf{0}}$ .
    - There can also be  $\underline{\mathbf{e}} \neq \underline{\mathbf{0}}$  that gives  $\underline{\mathbf{s}} = \underline{\mathbf{0}}$ .
      - For example, any nonzero  $\tilde{\underline{\mathbf{e}}} \in \mathcal{C}$ , will also give  $\underline{\mathbf{s}} = \underline{\mathbf{0}}$ .
      - However, they have larger weight than  $\underline{\mathbf{e}} = \underline{\mathbf{0}}$ .
    - The decoded codeword is the same as the received vector.
- Case 2: When,  $e_i = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$  (a pattern with a single one in the  $j^{\text{th}}$  position) we have  $\underline{\mathbf{s}} = \underline{\mathbf{v}}^{(j)} = (\text{the } j^{\text{th}} \text{ column of } \mathbf{H})^T$ 
  - When  $\underline{\mathbf{s}} = (\text{the } j^{\text{th}} \text{ column of } \mathbf{H})^T$ , we can conclude that  $\hat{e}_i = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$ 
    - There can also be other  $\underline{\mathbf{e}}$  that give  $\underline{\mathbf{s}} = \underline{\mathbf{v}}^{(j)}$ . However, their weights
      - can not be 0 (because, if so, we would have  $\underline{\mathbf{s}} = \underline{\mathbf{0}}$  but the columns of  $\mathbf{H}$  are nonzero)
      - nor 1 (because the columns of  $\mathbf{H}$  are distinct).
    - We flip the  $j^{\text{th}}$  bit of the received vector to get the decoded codeword.

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### 5.1 Binary Linear Block Codes

Interleaving

# Interleaving

- Conventional error-control methods such as parity checking are designed for **errors that are isolated or statistically independent events**.
- Some errors occur in **bursts** that span several successive bits.
  - Errors tend to group together in bursts. Thus, errors are no longer independent
  - Examples
    - **impulse noise** produced by lightning and switching transients
    - **fading** in wireless systems
    - channel with memory
- Such multiple errors wreak havoc on the performance of conventional codes and must be combated by special techniques.
- One solution is to spread out the transmitted codewords.
- We consider a type of interleaving called **block interleaving**.

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## Interleave as a verb

- To interleave = to combine different things so that parts of one thing are put between parts of another thing
- Ex. To interleave two books together:



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# Interleaving: Example

Consider a sequence of  $\ell$  blocks of coded data:

$$\left(x_1^{(1)} x_2^{(1)} \dots x_n^{(1)}\right) \left(x_1^{(2)} x_2^{(2)} \dots x_n^{(2)}\right) \dots \left(x_1^{(\ell)} x_2^{(\ell)} \dots x_n^{(\ell)}\right)$$



$$\begin{array}{cccc} x_1^{(1)} & x_2^{(1)} & \dots & x_n^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \dots & x_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(\ell)} & x_2^{(\ell)} & \dots & x_n^{(\ell)} \end{array}$$



$$\left(x_1^{(1)} x_1^{(2)} \dots x_1^{(\ell)}\right) \left(x_2^{(1)} x_2^{(2)} \dots x_2^{(\ell)}\right) \dots \left(x_n^{(1)} x_n^{(2)} \dots x_n^{(\ell)}\right)$$

- Arrange these blocks as rows of a table.
- Normally, we get the bit sequence simply by reading the table by rows.
- With interleaving (by an **interleaver**), transmission is accomplished by reading out of this table by columns.
- Here,  $\ell$  blocks each of length  $n$  are interleaved to form a sequence of length  $\ell n$ .

The received symbols must be deinterleaved (by a **deinterleaver**) prior to decoding.

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# Interleaving: Advantage

- Consider the case of a system that can only correct single errors.
- If an error burst happens to the original bit sequence, the system would be overwhelmed and unable to correct the problem.

original bit sequence  $\left(x_1^{(1)} x_2^{(1)} \dots x_n^{(1)}\right) \left(x_1^{(2)} x_2^{(2)} \dots x_n^{(2)}\right) \dots \left(x_1^{(\ell)} x_2^{(\ell)} \dots x_n^{(\ell)}\right)$

interleaved transmission  $\left(x_1^{(1)} x_1^{(2)} \dots x_1^{(\ell)}\right) \left(x_2^{(1)} x_2^{(2)} \dots x_2^{(\ell)}\right) \dots \left(x_n^{(1)} x_n^{(2)} \dots x_n^{(\ell)}\right)$

- However, in the interleaved transmission,
  - successive bits which come from **different** original blocks have been corrupted
  - when received, the bit sequence is reordered to its original form and then the FEC can correct the faulty bits
  - Therefore, single error-correction system is able to fix several errors.

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# Interleaving: Advantage

- If a burst of errors affects at most  $\ell$  consecutive bits, then each original block will have at most one error.
- If a burst of errors affects at most  $r\ell$  consecutive bits (assume  $r < n$ ), then each original block will have at most  $r$  errors.
- Assume that there are no other errors in the transmitted stream of  $\ell n$  bits.
  - A single error-correcting code can be used to correct a single burst spanning upto  $\ell$  symbols.
  - A double error-correcting code can be used to correct a single burst spanning upto  $2\ell$  symbols.

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